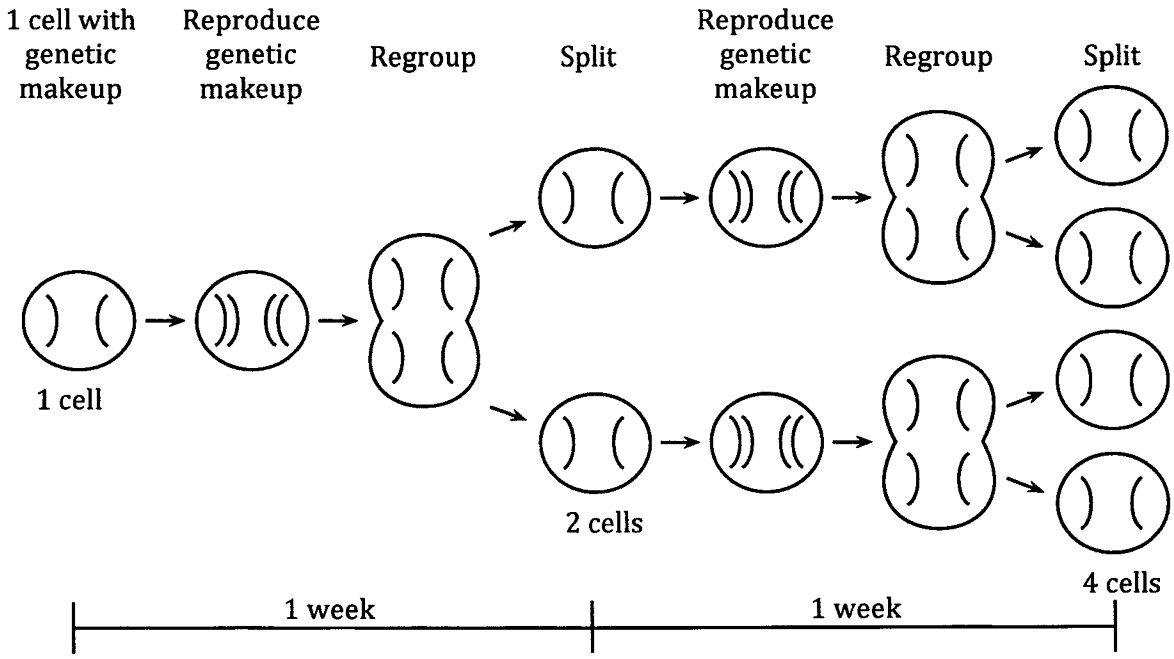
Chapter 2.

Exponential functions.

Situation One

The number of cells in a particular organism increases by cell division. In this process one cell splits into two cells which in turn each split into two cells and so on.

Let us suppose that there is initially 1 cell and that this cell division occurs approximately every week, i.e. the number of cells present doubles every week.



Copy and complete the following table.

Number of weeks, t weeks.	0	1	2	3	4	5	6	7	8	9	10
Number of cells, C.	1	2	4								

(a) How long does it take for there to be 100 cells present?

(b) How long does it take for there to be 400 cells present?

(c) Determine the rule for C in terms of t.

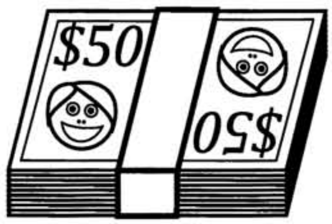
(d) Let us suppose that in a very simplified model of the growth of a human baby we assume that the initial single cell divides into two cells after one week. After another week these two each divide into two to give four cells altogether. If this cell division continues each week how many cells after 40 weeks?

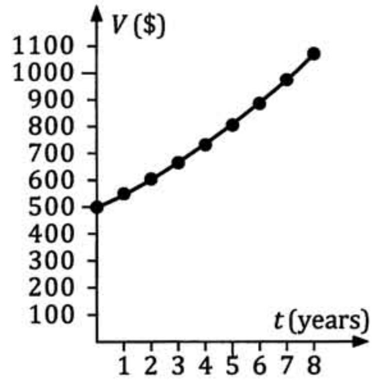
Now read Situations Two to Five that follow. They do not ask you to do any calculations. Simply read them and make sure that you agree with, and understand, what is said.

Situation Two.

Consider an investment of $500 earning interest of 10% compounded annually.
The value of this investment for the first eight years is shown tabulated below left and graphed below right.

Time (t yrs)	Value ($$V$)
0	$500
1	$550
2	$605
3	$665·50
4	$732·05
5	$805·26
6	$885·78
7	$974·36
8	$1071·79



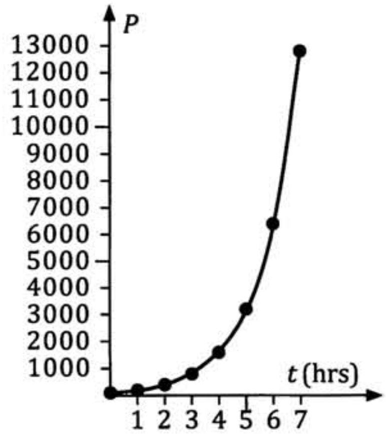


The figures in the V column commence with 500 and then each one thereafter is the previous one multiplied by 1·1.

Situation Three.

Consider a culture of bacteria with an initial population of 100 cells with the number doubling every hour.
This population for the first seven hours is shown tabulated below left and graphed below right.

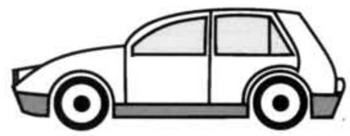
Time (t hrs)	Popul^n (P)
0	100
1	200
2	400
3	800
4	1600
5	3200
6	6400
7	12800



The figures in the P column commence with 100 and then each one thereafter is the previous one multiplied by 2.

Situation Four.

Consider a car that has an initial value of \$40 000 and by the end of each year it has lost 12% of what its value was at the beginning of that year. The table of values for the first seven years, the rule for the situation, and the graph, are shown below.

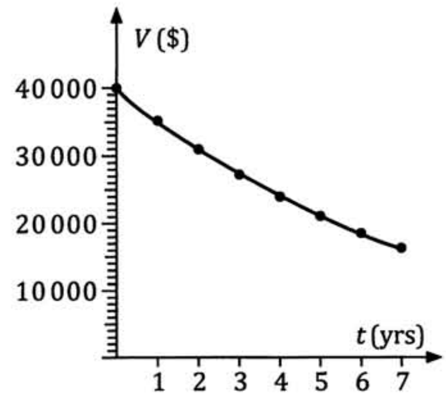


Time (t yrs)	Value (\$$V$)
0	\$40 000
1	\$35 200
2	\$30 976
3	\$27 259
4	\$23 988
5	\$21 109
6	\$18 576
7	\$16 347

Rule

Value after t years is given by :

$$V = 40\,000 \times 0{\cdot}88^t$$



Situation Five.

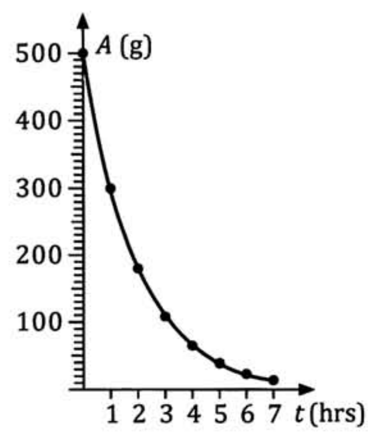
Consider a radioactive element decaying at a rate that sees 40% of the element decay to a more stable form each hour. Thus 500 g of the element becomes 300 g (= 60% of 500 g) one hour later, 180 g (= 60% of 300 g) one hour after that, and so on. The table of values for the first seven hours, the rule for the situation, and the graph, are shown below.

Time (t hrs)	Amount (A g)
0	500
1	300
2	180
3	108
4	65
5	39
6	23
7	14

Rule

Amount present after t hours is given by :

$$A = 500 \times 0{\cdot}6^t$$



Exponential relationships.

The situations on the previous pages involved a quantity being repeatedly multiplied by a number. The situations were all examples of **exponential relationships.**

In such relationships, because we repeatedly multiply by a number, the **ratio** of successive entries will be constant (rather than the first *difference pattern* being constant, as in linear, or the second *difference pattern* being constant, as in quadratic).

For example suppose we make a sequence of numbers for which the first is 3 and we repeatedly multiply by 2:

	3	6	12	24	48	96	192	
1st Difference	3	6	12	24	48	96		← Not const. Hence not linear.
Ratio	$\frac{6}{3}=2$	$\frac{12}{6}=2$	$\frac{24}{12}=2$	$\frac{48}{24}=2$	$\frac{96}{48}=2$	$\frac{192}{96}=2$		← Constant ratio. Exponential.

Each input value gives one and only one output value. The relationship is therefore a function. (Notice that the graphs on the previous pages pass the vertical line test.)

Exponential functions are characterised by **rules** of the form

$$y = y_0 a^x, \quad a > 0.$$

- y_0 is the value of y when $x = 0$,
- a is the constant multiplying factor.

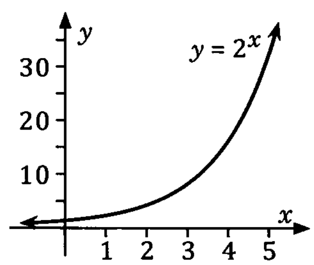
For example the rule $y = 3{\cdot}5 \times 4^x$ generates the following table of values:

x	0	1	2	3	4	5	6
y	3·5	14	56	224	896	3584	14336
1st ratio		$\frac{14}{3{\cdot}5}=4$	$\frac{56}{14}=4$	$\frac{224}{56}=4$	$\frac{896}{224}=4$	$\frac{3584}{896}=4$	$\frac{14336}{3584}=4$

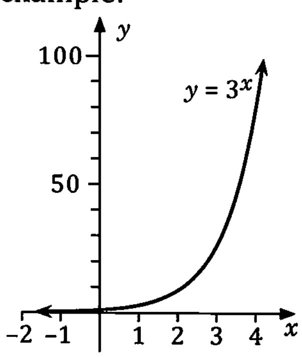
The **graphs** of exponential functions have the characteristic shape shown on the right by the graph of

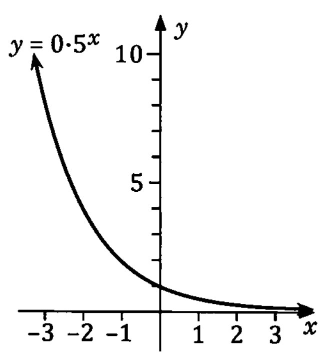
$$y = 2^x$$

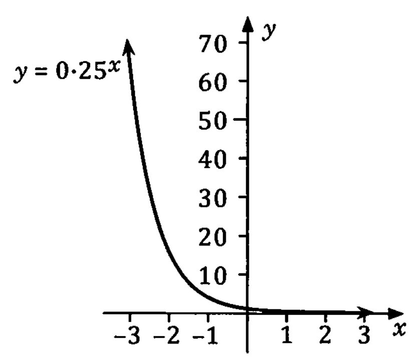
This characteristic shape will be reflected in the y-axis if the "a" in $y = a^x$ is such that $0 < a < 1$.)



For example:







Exercise 2A

For questions 1 to 6 copy and complete each of the following tables for the exponential rule stated.

1. Rule: $y = 3^x$

x	0	1	2	3	4	5
y						

2. Rule: $y = 7^x$

x	0	1	2	3	4	5
y						

3. Rule: $y = 1{\cdot}5 \times 2^x$

x	0	1	2	3	4	5
y						

4. Rule: $y = 1{\cdot}75 \times 8^x$

x	0	1	2	3	4	5
y						

5. Rule: $y = 2^{x+1}$

x	0	1	2	3	4	5
y						

6. Rule: $y = 2{\cdot}5 \times 4^{x+1}$

x	1	2	3	4	5	6
y						

For each of the tables shown in questions 7 to 18 below

(a) determine whether the function involved is linear, quadratic, cubic, exponential, reciprocal or none of these.

(b) For those that are one of the five types mentioned determine the equation of the function.

7.

x	0	1	2	3	4
y	1	2	5	10	17

8.

x	0	1	2	3	4
y	1	4	16	64	256

9.

x	0	1	2	3	4
y	3	5	7	9	11

10.

x	0	1	2	3	4
y	0	2	8	18	32

11.

x	0	1	2	3	4
y	1·5	12	96	768	6 144

12.

x	0	1	2	3	4
y	1	5	25	125	625

13.

x	0	1	2	3	4
y	0	2	6	12	20

14.

x	0	1	2	3	4
y	1	6	36	216	1296

15.

x	0	1	2	3	4
y	3	6	12	24	48

16.

x	1	2	3	4	5
y	60	30	20	15	12

17.

x	0	1	2	3	4
y	1	2	9	28	65

18.

x	0	1	2	3	4
y	20	17	14	11	8

19. (a) Display the graphs of the following exponential functions on a graphic calculator using an x-axis from 0 to 5 and a y-axis from 0 to 40.

$$y = 1{\cdot}25^x$$
$$y = 1{\cdot}5^x$$
$$y = 1{\cdot}75^x$$
$$y = 2^x$$
$$y = 3^x$$

State the coordinates of the point that all of these functions pass through.

(b) Write a few sentences, including sketches if you wish, to describe the characteristic shape of the graphs of functions of the form $y = a^x$ $(a > 1)$ and describe the effect changing the value of a has on the graph.

20. (a) Display the graphs of the following exponential functions on a graphic calculator using an x-axis from −3 to 4 and a y-axis from −1 to 10.

$$y = 2^x$$
$$y = 2(2)^x$$
$$y = 3(2)^x$$
$$y = 4(2)^x$$

(b) Write a few sentences, including sketches if you wish, to describe the effect increasing the value of a has on the graph of $y = a(2)^x$ for $a \geq 1$.

21. Investigate the effect changing the value of k has on the graph of

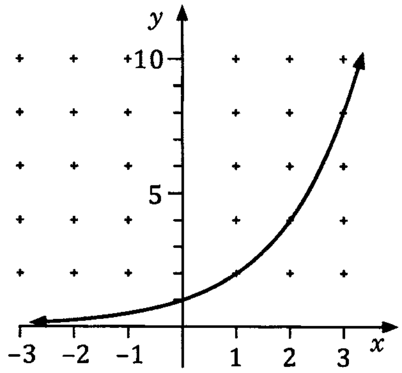
$$y = a^x - k.$$

22. Investigate the effect changing the value of k has on the graph of

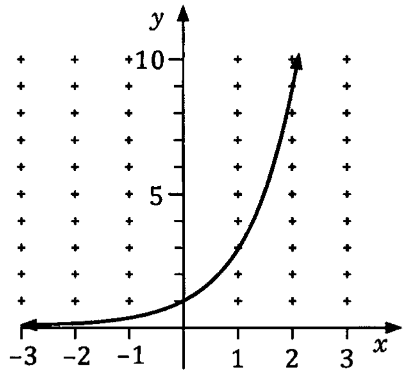
$$y = a^{x-k}.$$

23. Each graph shown below is of the form $y = a^x$, for integer a.
Find the equation of each.

(a)



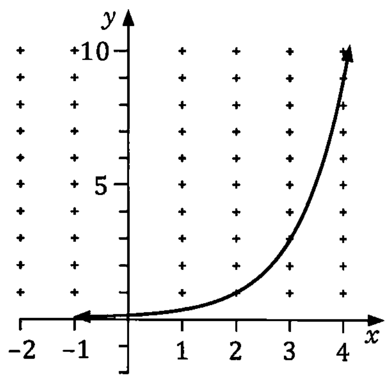
(b)



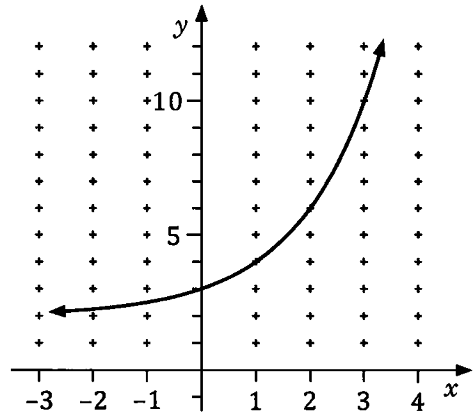
24. The population of a country is growing such that in t years' time the population will be P million where $P \approx 25\ (1{\cdot}04)^t$.
Display the graph of $P = 25\ (1{\cdot}04)^t$, for $0 \le t \le 50$, on a graphic calculator.
Using your graph, or by other methods, predict in how many years the population will be (a) 40 000 000, (b) 75 000 000, (c) 120 000 000.

25. Determine the equations of each of the exponential functions shown graphed below given that each is a *translation* of either $y = 2^x$ or $y = 3^x$.

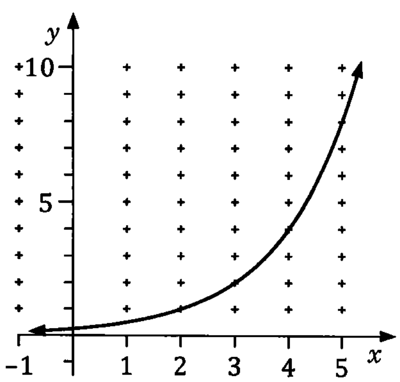
(a)



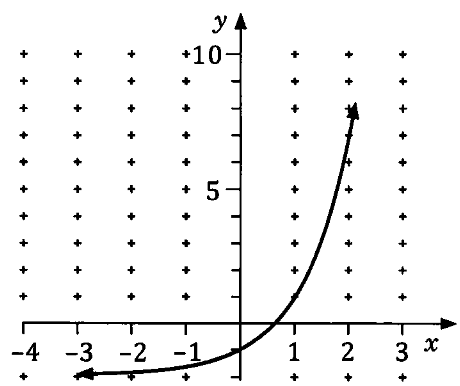
(b)



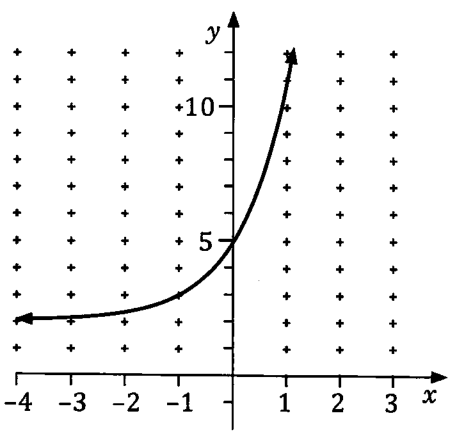
(c)



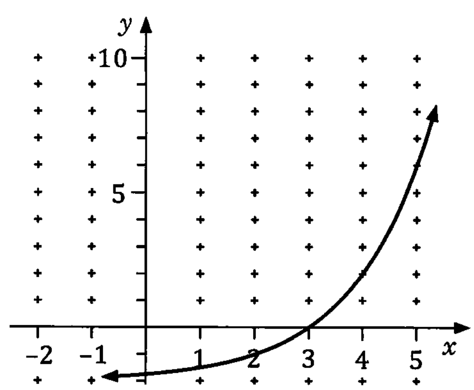
(d)



(e)



(f)



Growth and decay.

Thinking again about situations one to five at the beginning of this chapter:

Situation One, the cell splitting, involved a *growth* situation.	$C = 2^t$
Situation Two, the investment, was a *growth* situation.	$V = 500 \times 1{\cdot}1^t$
Situation Three, the culture of bacteria, was a *growth* situation.	$P = 100 \times 2^t$
Situation Four, the value of a car, was a *decay* situation.	$V = 40\,000 \times 0{\cdot}88^t$
Situation Five, the radioactive element, was a *decay* situation.	$A = 500 \times 0{\cdot}6^t$

Exponential *growth* is characterised by an equation of the form $y = y_0 a^x, \quad a > 1.$

Exponential *decay* is characterised by an equation of the form $y = y_0 a^x, \quad 0 < a < 1.$

Example 1

The population of a country is growing exponentially. The populations in 2010, 2011, 2012 and 2013 were as follows:

Year	2010	2011	2012	2013
Population	20 million	20·4 million	20·81 million	21·22 million

Predict the population for this country for the year 2030.

First find the annual growth rate:

$$\frac{20{\cdot}4}{20} = 1{\cdot}02$$

$$\frac{20{\cdot}81}{20{\cdot}4} \approx 1{\cdot}0201$$

$$\frac{21{\cdot}22}{20{\cdot}81} \approx 1{\cdot}0197$$

Each year the population is multiplied by roughly 1·02 (i.e. a 2% increase / year).

Thus by the year 2030 the population will be roughly $20 \text{ million} \times 1{\cdot}02^{(2030-2010)}$

$$= 20 \text{ million} \times 1{\cdot}02^{20}$$

$$\approx 29{\cdot}7 \text{ million}$$

The population for this country will be approximately 29·7 million by the year 2030 (assuming the growth rate shown in the given years continues).

Example 2

The population of a particular endangered species of animal is declining exponentially. The populations in 2000, 2005, 2010 and 2013 were thought to be as follows:

Year	2000	2005	2010	2013
Population	5760	4460	3450	2960

(a) By what percentage is the population declining each year?

(b) Predict the population for this animal for the year 2025.

(a) Consider the period 2000 to 2005.

If each year the population is multiplied by r then $5760r^5 = 4460$

$$\therefore \quad r^5 = \frac{4460}{5760}$$

$$\text{Thus} \quad r = \sqrt[5]{\frac{4460}{5760}} \approx 0{\cdot}95$$

Consider the period 2005 to 2010.

If each year the population is multiplied by r then $4460r^5 = 3450$

$$\therefore \quad r^5 = \frac{3450}{4460}$$

$$\text{Thus} \quad r = \sqrt[5]{\frac{3450}{4460}} \approx 0{\cdot}95$$

Thus each year the population is multiplied by 0·95 (i.e. a 5% decrease per year).

Check: For 2010 to 2013, $3450 \times 0{\cdot}95^3 \approx 2960$, as required.

The population is falling by 5% each year.

(b) By the year 2025 the population will be roughly $5760 \times 0{\cdot}95^{(2025-2000)}$

$$= 5760 \times 0{\cdot}95^{25}$$

$$\approx 1600$$

By the year 2025 the population will be roughly 1600 (assuming the rate of decline for the given years continues).

Example 3

The graph on the right shows an exponential growth situation with the variables A and t related according to a rule of the form

$$A = \text{k}\,\text{a}^t \text{ for a} > 1.$$

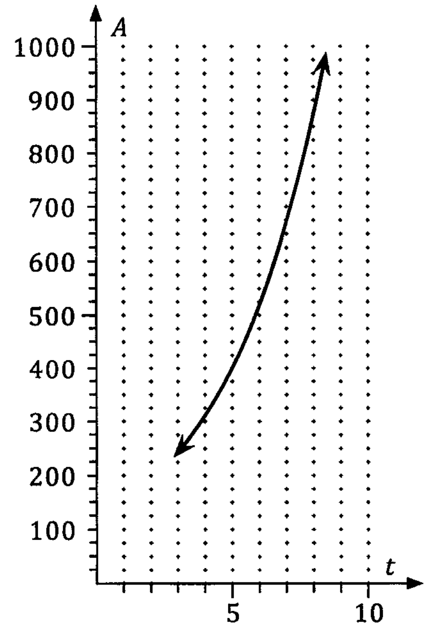
Determine (a) the value of A when $t = 5$,

(b) the value of A when $t = 8$,

(c) the constants a and k,

(d) the value of A when $t = 0$,

(e) the value of t (correct to one decimal place) for which $A = 4\,000$, assuming the growth rate suggested by the graph continues.



(a) From the graph, when $t = 5$, $A \approx 400$.

(b) From the graph, when $t = 8$, $A \approx 875$.

(c) Exponential growth is involved. Thus from $t = 5$ to $t = 8$ we have multiplied by "a" three times. Thus

$$\begin{aligned} 400\,\text{a}^3 &= 875 \\ \text{a} &= \sqrt[3]{2{\cdot}1875} \\ &\approx 1{\cdot}298 \end{aligned}$$

The relationship is of the form $A = \text{k}(1{\cdot}298)^t$

But when $t = 5$, $A \approx 400$. $\therefore\ 400 = \text{k}(1{\cdot}298)^5$

$$\text{k} \approx 109$$

Thus a $\approx 1{\cdot}298$ and k ≈ 109.

(d) The relationship is of the form $A \approx 109\,(1{\cdot}298)^t$

Thus when $t = 0$ $A \approx 109\,(1{\cdot}298)^0$

$$= 109$$

When t = 0, A $\approx$ 109.

(e) The relationship is of the form $A \approx 109\,(1{\cdot}298)^t$

Thus when $A = 4000$ $4000 \approx 109\,(1{\cdot}298)^t$

$$(1{\cdot}298)^t \approx 36{\cdot}70$$

Solving by calculator or by trial and adjustment

$$t = 13{\cdot}8, \text{ correct to one decimal place.}$$

Thus $A \approx 4000$ when $t = 13{\cdot}8$.

Note: With $A = \text{k}\text{a}^t$ then when $t = 0$, $A = \text{k}$. Thus had the graph in the above example shown where the curve cut the vertical axis this point would have given us the value of k directly.

Exercise 2B

1. Show that the following figures support the claim that the annual percentage growth rate is approximately 8%.

Year	1995	2000	2010
Population	18 000 000	26 000 000	56 000 000

2. Show that the following figures support the claim that the annual percentage decay rate is approximately 4·5%.

Year	2000	2007	2011
Population	12 400	9 000	7 500

3. The population of a country is growing exponentially. The populations in 2010, 2011, 2012 and 2013 were as follows:

Year	2010	2011	2012	2013
Population	45 million	45·8 million	46·6 million	47·5 million

Predict the population for this country for the year 2027.

4. The population of a particular species of animal is declining exponentially. The numbers of these animals thought to be in existence in the wild in 2010, 2011, 2012 and 2013 were as follows:

Year	2010	2011	2012	2013
Number	18 000	16 500	15 200	14 000

If the above figures are correct, and nothing is done to alter the rate of decline, how many of these animals will exist in the wild in the year 2023?

5. An analysis of the membership of a particular sports club since it was first formed in 1989 indicated that the membership each year could have been quite accurately predicted using the exponential model:

$$\text{Membership in the year } N \approx \mathrm{A}\mathrm{k}^{(N-1989)}.$$

The number of members initially and on the tenth, the twentieth and the twenty fifth anniversaries of the founding of the club were as follows:

Year	1989	1999	2009	2014
Members	80	170	375	550

(a) Find the values of A and of k (state k to 2 decimal places).
(b) By what percentage is the membership growing each year?
(c) Predict the number of members for the year 2024 (nearest hundred).

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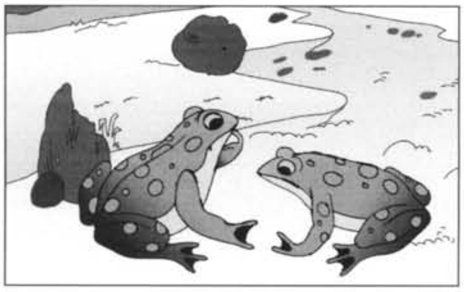
6. During a drought a particular river bed dries up and the colony of frogs living in the vicinity experiences an exponential population decline.
The estimated number of frogs in the colony, t days after drought conditions were officially declared, was as follows:

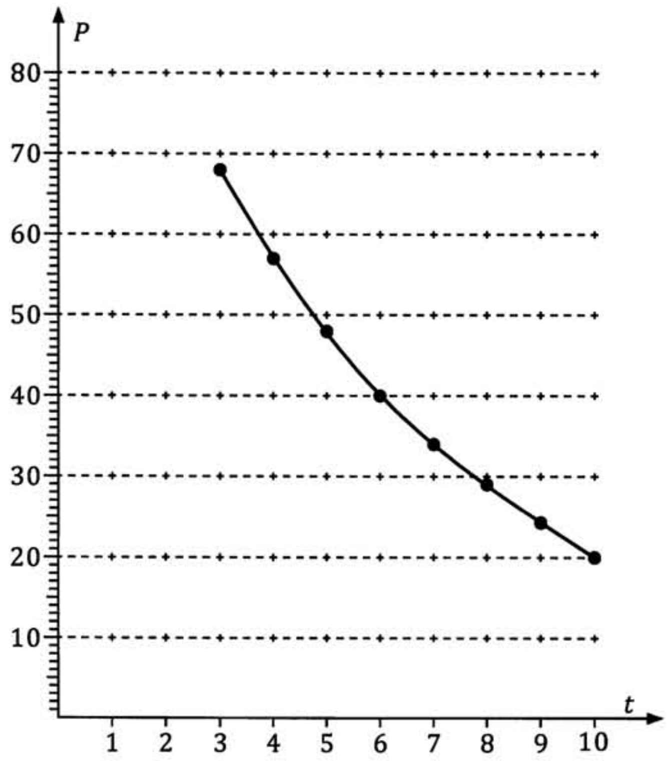


t	5	6	7	8
Population	530	450	385	325

According to these estimated figures what was the population of the frog colony initially (i.e. at $t = 0$).

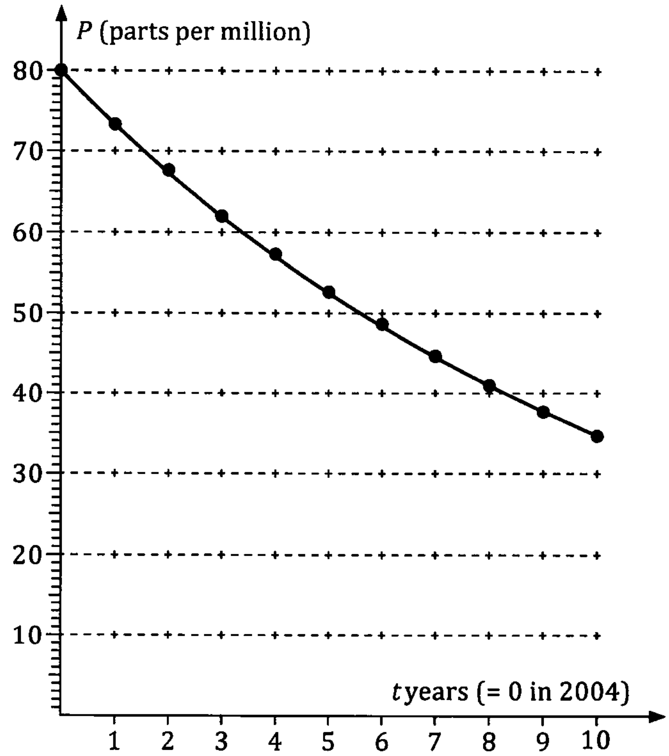
7. The graph below shows an exponential decay situation with the variables P and t related according to a rule of the form:

$$P = \mathrm{k}\,\mathrm{a}^t \quad \text{for} \quad 0 < \mathrm{a} < 1.$$



Determine
(a) the value of P when $t = 3$,
(b) the value of P when $t = 8$,
(c) the constants a (2 decimal places) and k (nearest 5),
(d) the value of P when $t = 0$ (nearest 5),
(e) the value of t (nearest integer) for which $P = 10$.

8. An environmental group commence a long term project to reduce the level of pollution in a particular stretch of river.
Starting the campaign in 2004 (when $t = 0$), the pollution level P, in parts per million, is monitored each year and the results are graphed as shown below.



The fall in P is thought to follow an exponential decline according to the rule

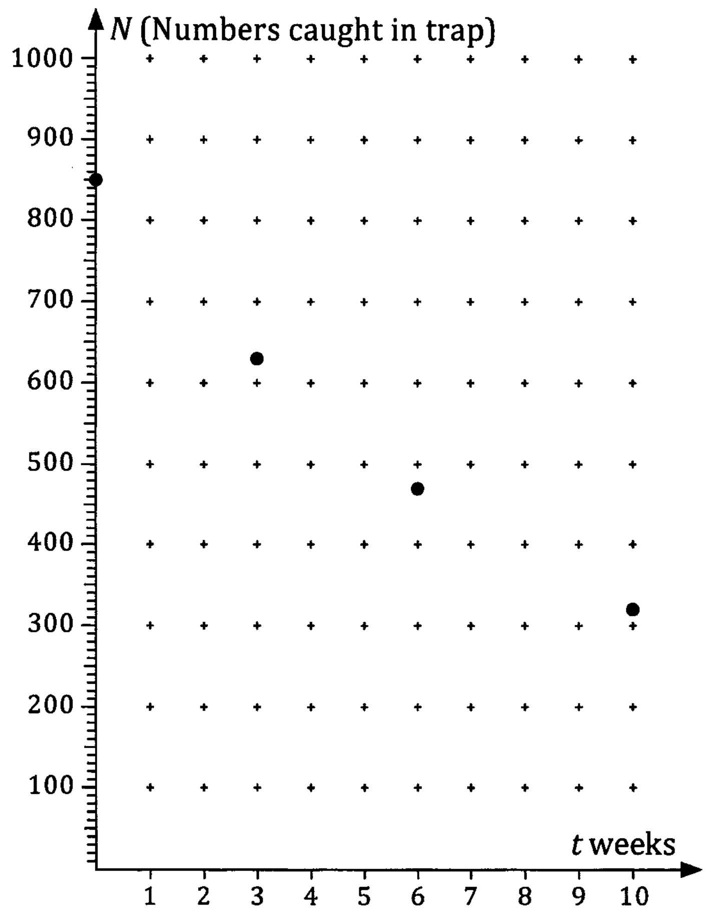
$$P = ka^t.$$

(a) Determine the value of k and of a.
(b) Use your formula to predict the value of P for 2017.
(c) The environmental group plan to release a number of fish into the river when they first record a value of P that is less than 20. When is this likely to be?

9. In a particular test area scientists note that when measures are introduced to reduce the population of a particular animal, animal A, classified as a pest, there is a rise in the population of another animal, animal B.
The scientists find that the decrease in the population of animal A and the increase in the population of animal B can both be modelled as exponential growth.
If P_A and P_B are the assessed populations of A and B respectively then t months after the introduction of control measures the populations are approximately given by: $P_A = 10\,000\,(0{\cdot}75)^t$ and $P_B = 1\,000\,(1{\cdot}09)^t$.

Find (a) the initial (i.e. $t = 0$) population of A and B in this test area,
(b) the population of A and B after 3 months of the control program (give answers correct to nearest 50).
(c) the value of t, correct to one decimal place, when the populations are equal.

10. To control an infestation of a certain flying insect in an area a number of sterile male insects are released into the area each week.
To monitor the effectiveness of the program traps are erected at the start of the program and again after 3, 6 and 10 weeks. Each time the traps are erected at the same time of the day, at the same place and for the same amount of time and the number of these insects caught is noted. The results are shown in the graph below.



(a) If the decline in the numbers caught is modelled by an exponential rule of the form $N = ka^t$, determine estimates for the constants k and a.

(b) The release of the sterile males will cease when the numbers caught in the traps is one quarter of the numbers caught initially (i.e. when $t = 0$).
After how many weeks is this likely to be?

Miscellaneous Exercise Two.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. For each of the following state which answer, I, II, III or IV shows the given number written in standard form, or scientific notation, i.e. in the form $A \times 10^n$ where A is a number between 1 and 10 and n is an integer.

		I	II	III	IV
(a)	36	$3{\cdot}6 \times 10^0$	$3{\cdot}6 \times 10^1$	$3{\cdot}6 \times 10^2$	$3{\cdot}6 \times 10^3$
(b)	0·000 023	$3{\cdot}2 \times 10^{-5}$	$2{\cdot}3 \times 10^5$	$2{\cdot}3 \times 10^{-4}$	$2{\cdot}3 \times 10^{-5}$
(c)	41 000	41×10^3	410×10^2	$4{\cdot}1 \times 10^4$	$4{\cdot}1 \times 10^{-4}$
(d)	0·245	$2{\cdot}45 \times 10^{-1}$	$0{\cdot}245 \times 10^0$	$2{\cdot}45 \times 10^{-2}$	$2{\cdot}45 \times 10^2$
(e)	0·003	3×10^{-2}	30×10^{-2}	3×10^{-3}	300
(f)	912 000	$9{\cdot}12 \times 10^4$	$9{\cdot}12 \times 10^7$	$9{\cdot}12 \times 10^6$	$9{\cdot}12 \times 10^5$
(g)	0·000 002 81	$2{\cdot}81 \times 10^5$	$2{\cdot}81 \times 10^{-5}$	$2{\cdot}81 \times 10^{-6}$	281×10^{-8}
(h)	14 200 000	$1{\cdot}42 \times 10^7$	$1{\cdot}42 \times 10^4$	$1{\cdot}42 \times 10^5$	$1{\cdot}42 \times 10^6$

2. Solve each of the following equations (without the assistance of a calculator).

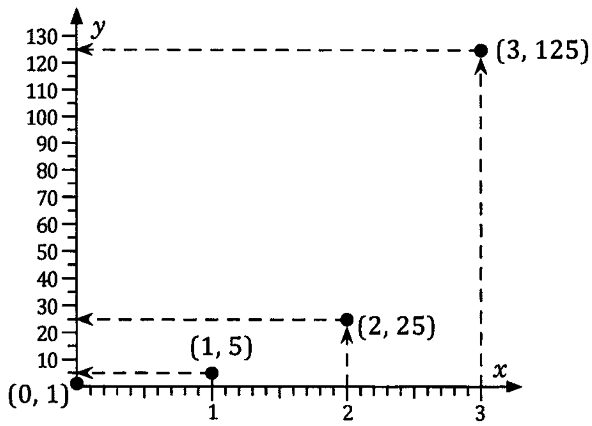
(a) $x^2 = 49$	(b) $x^2 = 100$	(c) $x^3 = 1000$
(d) $2^x = 4$	(e) $3^x = 81$	(f) $5^x + 11 = 12$
(g) $6^x + 9 = 225$	(h) $4^x = \frac{1}{4}$	(i) $4^x = \frac{1}{16}$
(j) $4^x = \frac{1}{64}$	(k) $2^x = 0{\cdot}5$	(l) $2^x = 0{\cdot}25$
(m) $2^x = 0{\cdot}125$	(n) $16x^4 = 400x^2$	(o) $8^{2x+1} = 4^{1-x}$
(p) $\sqrt{50}\,x - \sqrt{18}\,x = \sqrt{2}$	(q) $\sqrt{50x} - \sqrt{18x} = \sqrt{2}$	(r) $(x^3 + 5)(x^3 - 5) = 704$

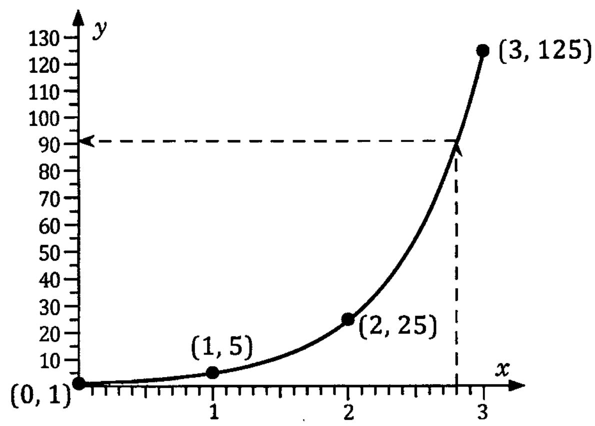
3. Round each of the following to the number of significant figures stated.

(a)	12 405	correct to two significant figures.
(b)	12 607 405	correct to four significant figures.
(c)	0·000 256	correct to two significant figures.
(d)	5·63	correct to one significant figures.
(e)	12 626·8	correct to four significant figures.

4. (a) What will be the equation of the graph obtained by translating the graph of the function $y = 2^x$ three units to the left, writing your answer both as $y = 2^{f(x)}$ and as $y = k \times 2^x$.
   (b) What will be the equation of the graph obtained by translating the graph of the function $y = 3^x$ two units down?

5. The graph below left shows four points which obey the rule $y = 5^x$.
Joining these points with a smooth curve, as shown below right, allows values for other powers of x to be suggested, e.g. $5^{2\cdot 8} \approx 91$.
Use the graph to suggest value for $5^{1\cdot 6}$, $5^{2\cdot 4}$ and $5^{2\cdot 5}$ and then check your answers using a calculator.

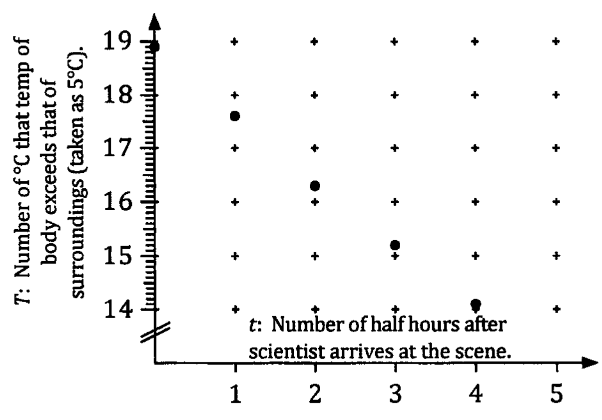




6. Without the assistance of a calculator, solve $(25 \times 5^x - 1)(5^x - 1) = 0$

7. Without the assistance of a calculator, solve $(2^x)^2 - 5(2^x) + 4 = 0$.

8. A forensic scientist is called to the scene of a murder. Upon arrival at 10 am the scientist notes the temperature of the body as being 23·9°C. This is 18·9°C above the temperature of the surrounding air (which is 5°C).
The scientist monitors this "body temperature above surrounding temperature of 5°C" at half hour intervals. The data collected is shown in the graph below.



(a) With T and t as defined in the graph and assuming the relationship between T and t is of the form $T = \mathrm{k}\,\mathrm{a}^t$, find the values of the constants k and a.

(b) If normal body temperature is 37°C, i.e. 32°C above the 5°C temperature of the surroundings, estimate the time of death.